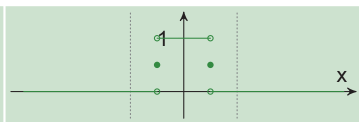
Pulse-Like Functions

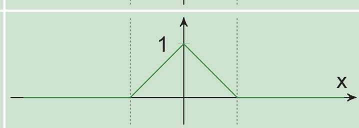
Pulse-like functions are localized in one or more independent dimensions. There are many processes in optics that are well modeled by these functions, including a burst of optical energy from a pulsed laser (temporal localization), the aperture of a lens (spatial localization), or a band-pass spectral filter (wavenumber localization). Furthermore, many periodic structures like diffraction gratings or lens arrays can be built up using a pulse-like function as the unit cell.

This *Field Guide* relies heavily on five pulse-like functions:

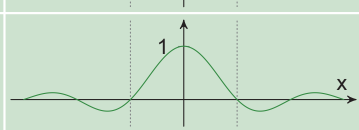
$$\mathrm{rect}(x) = \begin{cases} 1 & |x| < 1/2 \\ 1/2 & |x| = 1/2 \\ 0 & |x| > 1/2 \end{cases}$$



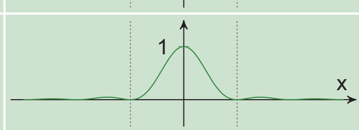
$$\mathrm{tri}(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| \geqslant 1 \end{cases}$$



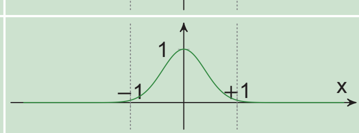
$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



$$\mathrm{sinc}^2(x) = \frac{\sin^2(\pi x)}{(\pi x)^2}$$



$$\mathrm{Gaus}(x) = e^{-\pi x^2}$$



The functions defined here are used throughout the literature with slightly different notation conventions. This *Field Guide* uses the selected definitions so that all functions are symmetric about $x = 0$ and integrate to unity:

$$\int_{-\infty}^{\infty} p(x)\, dx = 1$$

where $p(x)$ is any of the pulse functions defined above.

Causality

Causality is a property that is most widely applicable to time-domain systems. A causal system has no output until the input is active. Mathematically, this places the following limitation on the impulse response:

$$h(t) = 0 \ \forall t < 0$$

Causality is important for physically realizable transient systems. However, for functions of position, there is generally no practial importance to restricting the output for $x < 0$. For example, imaging systems with negative magnification produce outputs for $x < 0$, even if the input is zero for $x < 0$.

Kramers–Krönig Relationships

The requirement that $h(t) = 0$ for $t < 0$ for causal systems places restrictions on the relationship between the real and imaginary parts of the Fourier transform. An analytic function $h(t)$ is causal, and its Fourier transform is

$$\mathcal{F}\{f(t)\} = F(\nu) = F_r(\nu) + iF_i(\nu)$$

The real and imaginary parts of $F(\nu)$ satisfy

$$\begin{aligned} F_r(\nu) &= \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{F_i(\nu')}{\nu' - \nu} d\nu' \\ F_i(\nu) &= -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{F_r(\nu')}{\nu' - \nu} d\nu' \end{aligned}$$

The symbol $\mathcal{P}$ denotes the **Cauchy principal value** of the integral, which dictates how to deform the contour of integration due to the pole at $\nu' = \nu$.

> The Kramers–Krönig relationships place important restrictions on physical parameters of optical problems, such as the complex index of refraction of a medium. When dealing with an optical problem in the frequency domain, care must be taken to ensure that the Kramers–Krönig relationships are satisfied in order to obtain physically meaningful results.

Discrete Convolution

Define **discrete convolution** of two sequences $f * h$ to be

$$g_k = (f * h)_k = \sum_{m=-\infty}^{\infty} f_m h_{k-m}$$

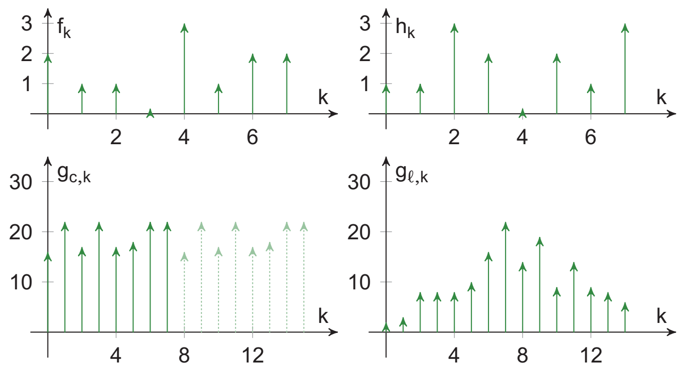
For finite sequences, summation limits can be narrowed to only accommodate the effective support of both sequences:

$$g_k = (f * h)_k = \sum_{m=-M}^{M} f_m h_{k-m}$$

An alternative way of calculating convolution is

$$G_n = F_n H_n \quad \rightarrow \quad g_k = \mathcal{D}^{-1}\{F_n H_n\}$$

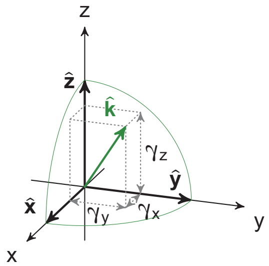
which is faster to compute given the optimizations available for DFT. The caveat is that the DFT method introduces an implied periodicity, and as a result, the beginning of one period may interact with the end of the other. This calculation is denoted as **circular convolution**, as opposed to the previously introduced **linear convolution**.



Two sequences of size N_1 and N_2 need to be padded with $\Delta N_1 = N_2 - 1$ and $\Delta N_2 = N_1 - 1$ zeroes, respectively, in order for circular and linear convolutions to produce the same results.

Plane Wave Spectrum

The direction of propagation of a plane wave is



$$\hat{\mathbf{k}} = \gamma_x \hat{\mathbf{x}} + \gamma_y \hat{\mathbf{y}} + \gamma_z \hat{\mathbf{z}}$$

where γ_x, γ_y, and γ_z are the **direction cosines** given by the cosine of the angle between $\hat{\mathbf{k}}$ and the coordinate axes, and they satisfy

$$\gamma_x{}^2 + \gamma_y{}^2 + \gamma_z{}^2 = 1$$

A plane wave with **wavenumber** $k = 2\pi/\lambda$ propagating along $\hat{\mathbf{k}}$ can be expressed as

$$u(x, y, z) = u_0 e^{ik(\gamma_x x + \gamma_y y + \gamma_z z)}$$

Consider the fields in the $z = 0$ plane written in terms of their Fourier transform:

$$u(x, y, z = 0) = \iint_{-\infty}^{\infty} U(\xi, \eta, z = 0) e^{i2\pi(\xi x + \eta y)} d\xi d\eta$$

The direction cosines can be expressed in terms of the spatial frequencies as

$$\gamma_x = \lambda\xi \qquad \gamma_y = \lambda\eta \qquad \gamma_z = \sqrt{1 - \lambda^2\left(\xi^2 + \eta^2\right)}$$

The inverse Fourier transform is just a superposition of plane waves propagating in different directions, which gives rise to the term **plane wave spectrum** or **angular spectrum**. For spatial frequencies

$$\lambda^2\left(\xi^2 + \eta^2\right) \leqslant 1$$

the direction cosine γ_z is purely real, and the plane wave is **propagating**. For spatial frequencies

$$\lambda^2\left(\xi^2 + \eta^2\right) > 1$$

the direction cosine γ_z is imaginary, and the plane wave is **evanescent**. Evanescent waves decay exponentially and can only be observed in the near field.

Channeled Spectropolarimetry

Polarimetry involves the measurement of the vector nature of the optical polarization signature. Because optical detectors do not generally respond to polarization, polarimeters operate by modulating the intensity of the light in a polarization-dependent fashion. An important class of polarimeters are the **channeled polarimeters** that accomplish this by creating FDM channels in space, time, wavenumber, angle, or some other modulation dimension.

Consider a beam of light $\mathbf{E} = u_x\hat{x} + u_y\hat{y}$. The Stokes parameters describing its polarization state are

$$\mathbf{S}(t) = \begin{bmatrix} s_0(t) \\ s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix} = \begin{bmatrix} \left\langle |u_x(t)|^2 \right\rangle + \left\langle |u_y(t)|^2 \right\rangle \\ \left\langle |u_x(t)|^2 \right\rangle - \left\langle |u_y(t)|^2 \right\rangle \\ 2\Re \left\langle u_x(t) u_y{}^*(t) \right\rangle \\ -2\Im \left\langle u_x(t) u_y{}^*(t) \right\rangle \end{bmatrix}$$

This beam is analyzed by a linear polarizer rotating at a constant angular velocity $\theta = 2\pi\nu_0 t$. This analyzer produces a time-varying irradiance

$$\begin{aligned} I(t) &= \frac{1}{2} \begin{bmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \end{bmatrix}^{\mathrm{T}} \cdot \mathbf{S}(t) \\ &= \frac{1}{2} \left[s_0(t) + s_1(t)\cos(4\pi\nu_0 t) + s_2(t)\sin(4\pi\nu_0 t) \right] \end{aligned}$$

The resulting irradiance is a **multiplexed signal** with the s_0 information carried in a channel centered at $\nu = 0$, the s_1 information in the real part of a channel centered at $\nu = 2\nu_0$, and the s_2 information in the imaginary part of a channel centered at $\nu = 2\nu_0$. More-complicated strategies have been developed that modulate in time, space, wavenumber, angle of incidence, and combinations of multiple independent variables.